Resit
of
Thermal Physics (T1-T6)
2020-2021

Saturday April 10, 2021

9.00-12.00

online

PROBLEM 1

Score: a+b =6+6=12

A container with a volume of 1.0 dm^3 contains $2.0 \cdot 10^{23}$ He atoms. The temperature of the He gas is 54.4 K. You can assume that the He behaves like an ideal gas.

Given: $M_{He} = 4.0$ g mol^{-1}, $M_{O_2} = 32.0$ g mol^{-1}.

a) Determine the gas pressure.
b) Draw a sketch of the Maxwell distribution. Roughly approximate in an oxygen gas the fraction of O_2 molecules that have molecular speeds in the range between 1500 m s^{-1} and 1600 m s^{-1} when the temperature is 300 K.

PROBLEM 2

Score: a+b+c=5+5+5=15

Consider a hypothetical engine which connects two (infinite) reservoirs and undergoes the following *reversible* processes:

i. isothermal expansion at temperature T_h
ii. adiabatic expansion to temperature T_c
iii. isothermal compression at temperature T_c
iv. adiabatic compression to the initial state

In every cycle, a quantity of heat Q_{in} flows from reservoir A into the engine and a quantity of heat Q_{out} flows from the engine into the reservoir B.

a) Sketch the thermodynamic cycle in a (T,S)-diagram. Indicate where the entropy of the engine changes and how these changes relate to the entropies of reservoirs A and B.
b) Use entropy arguments to show, that the efficiency of the cycle equals $\eta = 1 - T_c/T_h$.
c) The coal power plant in Eemshaven produces about 12 TWh of electricity per year from about 3 million tons of coal (thermal energy content of coal: 9 kWh kg^{-1}). Determine the efficiency of the power plant and estimate T_c and T_h assuming the power plant is an idealized Carnot engine. Is the result for T_h realistic? Estimate the efficiency of a primitive steam engine (use realistic numbers) and compare.

PROBLEM 3

Score: a =6

The thermal diffusion equation for a sphere can be written as,

$$\frac{\partial T}{\partial t} = D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$$

if there is no dependence on the angular coordinates. In this equation r is the radial coordinate, D is the diffusion coefficient and T is the temperature.

a) Give a general solution for T for the steady state case.

PROBLEM 4

Score: a+b=6+6=12

Typical turbomolecular pumps can generate a vacuum of about $1\cdot 10^{-10}$ bar. Assume the pump to be at $T = 25°\,C$ and working on a gas primarily consisting of N_2 molecules.

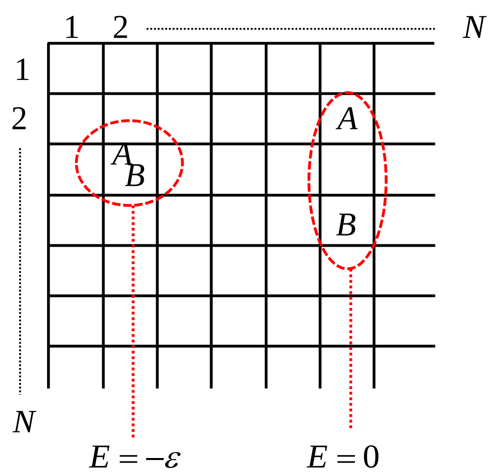
a) Calculate the mean free path of the molecules and the collision frequency.
b) Do the molecules that pass the turbomolecular pump have the same Maxwell Boltzmann distribution of speeds that the gas in the vacuum chamber has? (Hint: assume that the pump has a diameter of 10 cm. Use the result for the mean free path to argue.) If the answer is no, what speed distribution do these molecules have?

Collision diameter N_2: 395 pm.

PROBLEM 5

Score: a+b+c =5+4+4=13

A 2-dimensional lattice is in equilibrium with a heath bath at temperature T. The lattice has $N \times N$ lattice positions (see figure). On this lattice there are two distinguishable particles A and B that can freely move over the lattice positions. In the situation that the particles are at the same lattice position the energy of the system of two particles is $-\varepsilon$; in the situation that the particles are at different lattice positions this energy is zero.



a) How many possibilities (microstates) are there to place the two particles on the lattice? How many of these microstates have energy $-\varepsilon$ and how many have energy zero.

b) Give the general expression of the partition function Z when you sum over all different *energies* of a system. Use this to show that the partition function Z for the system of the two particles on the lattice is:

$$Z = N^2\left(N^2 - 1 + e^{\beta\varepsilon}\right)$$

c) Use this partition function to calculate the internal energy U of the system of the two particles on the lattice.

PROBLEM 6

Score: a+b+c=6+6+5=17

The Berthelot equation of state is given by:

$$P = \frac{RT}{V-b} - \frac{a}{TV^2}$$

in which P, V, T are the pressure, the *molar* volume and the temperature of the gas, respectively. The constant a controls the attractive molecular interactions and the constant b corrects for the volume of the gas molecules.

a) Calculate the second virial coefficient $B(T)$ of the Berthelot gas.
b) Calculate the Boyle temperature T_b of the Berthelot gas.

A crude model for the intermolecular potential is the square well potential. Suppose that for a certain real gas (not necessarily a Berthelot gas) we have the following square well potential $v_{\kappa,R,\varepsilon}(r)$ describing the interaction between two molecules as a function of their separation distance r:

$$v(r) = \infty; \qquad 0 < r \leq \frac{R}{\kappa}$$

$$v(r) = -\varepsilon; \qquad \frac{R}{\kappa} < r \leq R$$

$$v(r) = 0; \qquad r > R$$

with κ a dimensionless constant such that $\kappa > 1$, ε has the units of energy and the radius R is expressed in units of length.

c) Calculate the second virial coefficient $B(T)$ (per mole) for a real gas with such a square well potential $v_{\kappa,R,\varepsilon}(r)$. Express your answer in terms of R, ε, κ and β.

PROBLEM 7

Score: a+b+c =6+5+4=15

Consider a one-atom layer thick square (with sides of length L) of metallic atoms. The square consists of N atoms of which each atom contributes exactly *two* electrons to the total amount of conduction electrons. These conduction electrons can be considered as a 2D ideal gas of fermions with spin $\frac{1}{2}$ enclosed in a square with area $A = L^2$.

a) Show that number of states $\Gamma(E)$ with energy smaller than E for this 2D gas of fermions is proportional to E:

$$\Gamma(E) = \frac{AE}{\sigma}$$

with σ a constant. Give an expression for σ in terms of fundamental constants.

Use the expression for $\Gamma(E)$ to show that the density of states $g(E)dE$ for this 2D ideal gas of electrons is independent of energy and can be written as:

$$g(E)dE = \frac{AdE}{\sigma}$$

We now cool the square of metallic atoms to temperature $T = 0$.

b) Calculate the Fermi energy E_F for this 2D ideal gas of electrons. Express your answer in N, A and σ.

c) Show that at $T = 0$ the internal energy of this gas is given by $U = NE_F$.

Solutions

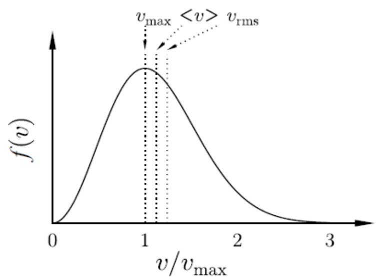
PROBLEM 1

a)

$$pV = Nk_BT$$

$$p = \frac{Nk_BT}{V} = \frac{2 \times 10^{23} \times 1.38\ \times 10^{-23}\ \mathrm{J}/\mathrm{K} \times 54.4\ \mathrm{K}}{10^{-3}\mathrm{m}} \approx 150\ \mathrm{kPa}$$

b)



$$f(v) = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2k_BT}\right)^{3/2} v^2 dv\ e^{-\frac{mv^2}{2k_BT}}$$

$$m = \frac{0.032\ \mathrm{kg\ mol}^{-1}}{N_A} \approx 5.3 \times 10^{-26}\mathrm{kg}$$

$$f(1550\ \mathrm{ms}^{-1}) = \frac{4}{\sqrt{\pi}}\left(\frac{5.3 \cdot 10^{-26}\mathrm{kg}}{2k_B 300\ \mathrm{K}}\right)^{3/2} (1050\ \mathrm{ms}^{-1})^2 dv\ e^{-\frac{5.3 \cdot 10^{-26}\mathrm{kg}(1050\ \mathrm{ms}^{-1})^2}{2k_B 300\mathrm{K}}}$$

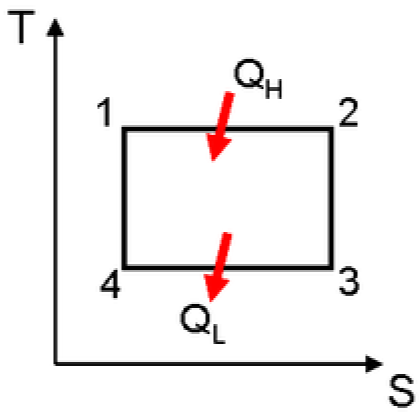
$$\approx 1.2 \cdot 10^{-4}\ \mathrm{sm}^{-1} dv$$

$$\Delta v = (1500 - 1600)\mathrm{ms}^{-1} = 100\ \mathrm{ms}^{-1}$$

The fraction is approximately 0.012%.

PROBLEM 2

a)



Entropy changes in the isothermal steps i) and iii) (heat flows to keep T constant). Adiabatic means no heat flow, i.e. no entropy change.

$$\Delta S_A = \int \frac{dQ}{T_h} = \frac{\Delta Q}{T_h} = \frac{Q_h}{T_h}$$

$$\Delta S_B = \frac{Q_c}{T_c}$$

b)

$$\Delta S_h = \Delta S_c$$

$$\frac{Q_h}{T_h} = \frac{Q_c}{T_c}$$

$$\eta = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h} = 1 - \frac{T_c}{T_h}$$

c)

$$\eta = \frac{12\ \text{TWh}}{3 \cdot 10^9\text{kg} \times 9\ \text{kWh kg}^{-1}} \approx 0.44$$

The cooling water is liquid water from outside the plant (e.g. the sea), let's say $T_c = 300$ K.

$$\eta = 1 - \frac{T_c}{T_h}$$

$$T_h \approx \frac{T_c}{0.56} = 535\ \text{K}$$

This is for an idealized Carnot process. The actual temperatures are much higher!

Steam engine: T_h=100°C=373K, T_c=50 °C=323K, $\eta = 1 - \frac{T_c}{T_h} \approx 0.1$

PROBLEM 3

a)

$$D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = 0$$
$$r^2\frac{\partial T}{\partial r} = const$$
$$T(r) = A + \frac{B}{r}$$

PROBLEM 4

a)

$$\lambda = \frac{1}{\sqrt{2}n\sigma}$$

$$n = \frac{N}{V} = \frac{p}{k_B T} = \frac{1 \times 10^{-5}\mathrm{Pa}}{1.38 \times \frac{10^{-23}\mathrm{J}}{\mathrm{K}} \times 298\ \mathrm{K}} \approx 2.4 \times 10^{15}\mathrm{m}^{-3}$$
$$\sigma = \pi \times (395 \times 10^{-12})^2 = 4.9 \times 10^{-19}\mathrm{m}^2$$
$$\lambda = \frac{1}{\sqrt{2} \times 2.4 \times 10^{15}\mathrm{m}^{-3} \times 4.9 \times 10^{-19}\mathrm{m}^2} = 601\ \mathrm{m}$$

This is huge, because of the low pressure, molecules essentially do not collide with each other in a normal-sized container.

b) Despite the macroscopic diameter of the pump, we are dealing with effusion rather than with a regular Maxwell-Boltzmann distribution, simply because the mean free path is large as compared to the diameter of the opening.

PROBLEM 5

a)

Total number of microstates: $N \times N$ possibilities for the first particle times $N \times N$ possibilities for the second particle gives N^4 microstates. The number of microstates with energy $E = -\varepsilon$ is equal to the number of lattice positions: N^2. The number of microstates that have energy $E = 0$ is $N^4 - N^2 = N^2(N^2 - 1)$.

b)

General expression for the partition function:

$$Z = \sum_r e^{-\beta E_r}$$

where the summation is over all microstates r or

$$Z = \sum_{E_r} g(E_r) e^{-\beta E_r}$$

where the summation is over all different energies E_r and $g(E_r)$ is the degeneracy of the energy E_r (number of microstates with that energy).

For the system of the two particles on the lattice:

$$Z = N^2 e^{-\beta(-\varepsilon)} + N^2(N^2 - 1)e^{-\beta 0} = N^2(N^2 - 1) + N^2 e^{\beta\varepsilon} = N^2\left(N^2 - 1 + e^{\beta\varepsilon}\right)$$

c)

We use $U = -\frac{\partial \ln Z}{\partial \beta}$ and find,

$$U = -\frac{N^2 \varepsilon e^{\beta\varepsilon}}{N^2(N^2 - 1) + N^2 e^{\beta\varepsilon}} = \frac{-\varepsilon e^{\beta\varepsilon}}{(N^2 - 1) + e^{\beta\varepsilon}}$$

PROBLEM 6

a)
Rewrite the Berthelot equation as:

$$\frac{PV}{RT} = \frac{1}{\left(1 - \frac{b}{V}\right)} - \frac{a}{RT^2V}$$

and expand the first term on the right-hand side in powers of $\frac{1}{V}$:

$$\frac{PV}{RT} = \left(1 + \frac{b}{V} + \left(\frac{b}{V}\right)^2 + \cdots\right) - \frac{a}{RT^2V} \Rightarrow$$

$$\frac{PV}{RT} = 1 + \left(b - \frac{a}{RT^2}\right)\frac{1}{V} + \cdots$$

Thus,

$$B(T) = b - \frac{a}{RT^2}$$

b)
The temperature at which the second virial coefficient is zero is called the Boyle temperature.

$$B(T) = 0 \Rightarrow b - \frac{a}{RT^2} = 0 \Rightarrow T_b = \sqrt{\frac{a}{bR}}$$

At this temperature Boyle's law $(PV = constant)$ approximately holds for a real gas.

c)

$$B(T) = \frac{N}{2}\int \left(1 - e^{-\beta v(r)}\right) d^3r$$

$$= \frac{N}{2}\int_0^{\frac{R}{\kappa}} 4\pi r^2 dr + \frac{N}{2}\int_{\frac{R}{\kappa}}^{R} \left(1 - e^{\beta\varepsilon}\right) 4\pi r^2 dr + \frac{N}{2}\int_R^{\infty} 0 \cdot 4\pi r^2 dr \Rightarrow$$

$$\frac{B(T)}{N} = 2\pi \int_0^{\frac{R}{\kappa}} r^2 dr + 2\pi\left(1 - e^{\beta\varepsilon}\right) \int_{\frac{R}{\kappa}}^{R} r^2 dr + 0$$

$$= \frac{2\pi}{3}\left(\left(\frac{R}{\kappa}\right)^3 + \left(1 - e^{\beta\varepsilon}\right)\left(R^3 - \left(\frac{R}{\kappa}\right)^3\right)\right) \Rightarrow$$

$$\frac{B(T)}{N} = \frac{2\pi}{3}\left(R^3 - e^{\beta\varepsilon}\left(R^3 - \left(\frac{R}{\kappa}\right)^3\right)\right) = \frac{2\pi}{3} R^3 \left(1 - e^{\beta\varepsilon}\left(\frac{\kappa^3 - 1}{\kappa^3}\right)\right)$$

PROBLEM 7

a)

From the solution of the 2D-wave equation: $\varphi = A \sin k_x x \sin k_y y$ and taking this function to vanish at $x = y = 0$ and at $x = y = L$ results in,

$k_x = \frac{n_x \pi}{L}$ and $k_y = \frac{n_y \pi}{L}$ with n_x and n_y non-zero positive integers.

The total number of states with $|\vec{k}| < k$ is then given by, (the area of a quarter circle because we have only positive integers, with radius k divided by the area of the unit surface e.g. the surface of one state, in k-space).

$$\Gamma(k) = \frac{\frac{1}{4}\pi k^2}{\left(\frac{\pi}{L}\right)^2} = \frac{1}{4}\frac{L^2 k^2}{\pi}$$

Converting to energy $p = \sqrt{2mE} = \hbar k$ we find, $k = \frac{\sqrt{2mE}}{\hbar}$ and $dk = \frac{1}{2}\frac{2m}{\hbar\sqrt{2mE}} dE$

We find,

$$\Gamma(E) = \frac{1}{4}\frac{L^2 k^2}{\pi} = \frac{1}{4}\frac{L^2}{\pi}\frac{2mE}{\hbar^2} = \frac{1}{2}\frac{A}{\pi}\frac{mE}{\hbar^2}$$

For fermions with spin $\frac{1}{2}$ we have two spin states thus,

$$\Gamma(E) = 2 \times \frac{1}{2}\frac{A}{\pi}\frac{mE}{\hbar^2} = \frac{A}{\pi}\frac{mE}{\hbar^2} = \frac{AE}{\sigma}$$

Thus, $\sigma = \frac{\pi\hbar^2}{m}$

The number of states between $E + dE$ and E is:

$$g(E)dE = \Gamma(E + dE) - \Gamma(E) = \frac{\partial \Gamma}{\partial E} dE = \frac{AdE}{\sigma}$$

b)

The Fermi energy is the value of the chemical potential μ at absolute zero temperature:

$$E_F = \mu(T = 0)$$

Total number of fermions is given by,

$$2N = \int_0^\infty n(E)\, g(E) dE$$

with,

$$n(E) = \frac{1}{e^{\beta(E-\mu)} + 1}$$

the mean number of fermions with energy E (Fermi-Dirac distribution)

Thus,

$$2N = \frac{A}{\sigma} \int_0^\infty \frac{dE}{e^{\beta(E-\mu)} + 1}$$

And at $T = 0$, we have $E_F = \mu(T = 0)$ and thus $n(E) = 1$ if $E < E_F$ and $n(E) = 0$ if $E > E_F$. Thus,

$$2N = \frac{A}{\sigma} \int_0^{E_F} dE = \frac{A}{\sigma} E_F \Rightarrow E_F = \frac{2N\sigma}{A}$$

c)

$$U = \int_0^\infty E n(E)\, g(E) dE = \frac{A}{\sigma} \int_0^\infty \frac{E dE}{e^{\beta(E-\mu)} + 1} \underset{T=0}{\Longrightarrow} U = \frac{A}{\sigma} \int_0^{E_F} E dE = \frac{A}{2\sigma} E_F^2 = N E_F$$